MATH 4030 Differential Geometry
Problem Set 4
due 3/11/2017 (Fri) at 5PM

Problems

(to be handed in)

Unless otherwise stated, we use U, O to denote connected open subsets of $\mathbb{R}^n$. The symbols S, S_1, S_2, S_3 always denote surfaces in $\mathbb{R}^3$. Recall that a *critical point* of a smooth function $f : S \to \mathbb{R}$ is a point $p \in S$ such that $df_p = 0$.

1. (a) Let p be a point on a surface $S \subset \mathbb{R}^3$. Prove that $K(p) > 0$ if and only if there exists a point $p_0 \in \mathbb{R}^3$ such that p is a local maximum of the function $f(x) = |x - p_0|^2$.
   (b) Show that there is no compact surface $S \subset \mathbb{R}^3$ with $K \leq 0$ everywhere.

2. Compute the mean curvature H and Gauss curvature K of the *catenoid* and the *helicoid* given respectively by the parametrizations

$$X(u, v) = (\cosh v \cos u, \cosh v \sin u, v), \qquad (u, v) \in (0, 2\pi) \times \mathbb{R},$$

$$X(u, v) = (v \cos u, v \sin u, u), \qquad (u, v) \in \mathbb{R}^2.$$

3. Show that a graphical surface $S = \{z = f(x, y)\}$ is *minimal* (i.e. $H \equiv 0$) if and only if f satisfies the *minimal surface equation*:

$$(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + (1 + f_y^2) f_{xx} = 0.$$

4. Compute the mean curvature H and Gauss curvature K of the torus of revolution given by the parametrization

$$X(u, v) = ((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u), \qquad (u, v) \in (0, 2\pi) \times (0, 2\pi),$$

   where $a > b > 0$ are constants.

5. Let $S_1 = \{z = 0\}$ be the xy-plane and $S_2 = \{x^2 + y^2 = 1\}$ be the right unit cylinder. Show that the map $f : S_1 \to S_2$ defined by $f(x, y, 0) = (\cos x, \sin x, y)$ is a local isometry.

6. Find a local isometry $f : S_1 \to S_2$ from the upper half plane $S_1 = \{z = 0, y > 0\}$ to the cone $S_2 := \{x^2 + y^2 = z^2, z > 0\}$. Calculate the mean and Gauss curvatures of S_2.

7. Find a local isometry between the helicoid and the catenoid. Are they globally isometric?

Suggested Exercises

(no need to hand in)

1. Let $S \subset \mathbb{R}^3$ be a surface. Fix $p_0 \in \mathbb{R}^3$ and consider the smooth function $f : S \to \mathbb{R}$ defined by $f(p) = |p - p_0|^2$.
   (a) Show that $p \in S$ is a critical point of f if and only if p_0 lies on the *normal line* of S at p, i.e.
   $$p_0 = p + \lambda N(p)$$
   for some $\lambda \in \mathbb{R}$, here $N(p)$ is *any* unit normal to S at p.
   (b) Calculate the hessian of f at a critical point $p \in S$ and show that
   $$d^2 f_p(v) = 2\big(|v|^2 - \lambda A(v, v)\big),$$
   where λ is the constant as in (a) and A is the second fundamental form of S at p (relative to the normal N as in (a)).

2. Describe the region of $\mathbb{S}^2$ covered by the image of the Gauss map of the following surfaces:
   (a) Paraboloid of revolution $z = x^2 + y^2$.
   (b) Hyperboloid of revolution $x^2 + y^2 - z^2 = 1$.
   (c) Catenoid $x^2 + y^2 = \cosh^2 z$.

3. Determine all the umbilic points on the ellipsoid (where $a > b > c > 0$ are distinct positive real numbers):
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
   At which point(s) does the Gauss curvature K attains its maximum? What about for the mean curvature H (assuming the orientation is chosen such that $H > 0$)?

4. Let $f : S_1 \to S_2$ be an isometry between two compact surfaces S_1, S_2 in $\mathbb{R}^3$. Show that S_1 and S_2 have the same area. *(You can assume that S_2 is covered by a single parametrization except a set of measure zero.)*

5. Suppose $X : U \to V \subset S$ and $\tilde{X} : U \to \tilde{V} \subset \tilde{S}$ are parametrizations of two surfaces $S, \tilde{S}$ in $\mathbb{R}^3$ such that their first fundamental forms are the same, i.e. for $i, j = 1, 2$,
   $$g_{ij} = \langle \frac{\partial X}{\partial u_i}, \frac{\partial X}{\partial u_j} \rangle = \langle \frac{\partial \tilde{X}}{\partial u_i}, \frac{\partial \tilde{X}}{\partial u_j} \rangle = \tilde{g}_{ij} \qquad \text{on } U.$$
   Show that V is isometric to $\tilde{V}$.

6. Given a surface $S \subset \mathbb{R}^3$, prove that the set of isometries $f : S \to S$ form a group under composition. This is called the *isometry group of S*. What is the isometry group of the unit sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$?